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ON THE PRODUCT OF UPPER IRREDUNDANCE NUMBERS OF A GRAPH AND ITS COMPLEMENT

E.J. COCKAYNE
University of Victoria, Victoria, B.C. Canada

and

C.M. MYNHARDT
University of South Africa, Pretoria, South Africa

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A set X of vertices of a graph is irredundant if the closed neighbourhood of each $x \in X$ is not contained in the union of closed neighbourhoods of the vertices of $X - \{x\}$. The upper irredundance number, $\mathrm{IR}(G)$ is the largest number of vertices in any irredundant set of G. We prove that for any p-vertex graph G, $\mathrm{IR}(G) \cdot \mathrm{IR}(\bar{G}) \leq \left\lceil \frac{p(p+2)}{4} \right\rceil$ and exhibit all graphs which attain this bound.

1. Introduction

The vertex x in a subset X of vertices of a graph is called *redundant in X* if its closed neighbourhood is contained in the union of closed neighbourhoods of the vertices of $X - \{x\}$. The set X is called *irredundant* if it contains no vertex which is redundant in X. We will need the following equivalent definition in which $N_G(x)$ denotes the open neighbourhood of vertex x in the graph $G = (V, E)$, and $G[X]$ denotes the subgraph of G induced by X.

The set X is irredundant in G if and only if for each $x \in X$, either

(i) x is an isolated vertex of $G[X]$, or

(ii) there exists a vertex x' such that

$$x' \in N_G(x) \cap (V - X) \quad \text{and} \quad N_G(x') \cap X = \{x\}. \tag{1}$$

The concept of irredundance was introduced originally in [3]. It is closely related to domination and independence in graphs. An excellent bibliography of existing results is given in [4].

The upper irredundance number of G, denoted by $\mathrm{IR}(G)$, is the maximum number of vertices in an irredundant set of G. Some results concerning this parameter were established in [2]. In this paper, we prove that for any p-vertex graph G, $\mathrm{IR}(G) \cdot \mathrm{IR}(\bar{G}) \leq \left\lceil \frac{p^2+2p}{4} \right\rceil$ and exhibit all the graphs which attain this upper bound. This theorem may be called a Nordhaus and Gaddum type result in reference to the work of these authors on the product of chromatic numbers of G and $\bar{G}$ (see [6]).

2. The result

Theorem 1. (a) *For any p-vertex graph* $G=(V, E)$,

$$\mathrm{IR}(G)\cdot\mathrm{IR}(\bar{G})\leqslant\left\lceil\frac{p^2+2p}{4}\right\rceil.$$

(b) *G attains the bound in* (a) *if and only if G or its complement consists of*

(i) *A set X of* $\left\lfloor\frac{p+1}{2}\right\rfloor$ *independent vertices,*

(ii) *A set Y of* $\left\lceil\frac{p+1}{2}\right\rceil$ *vertices where* $G[Y]$ *is complete and* $X\cap Y=\{x\}$, *and*

(iii) *any set S of edges joining vertices of* $X-\{x\}$ *to vertices of* $Y-\{x\}$.

Proof (a). Let X, Y be irredundant sets of G, $\bar{G}$ respectively where $|X|=m$ and $|Y|=n$. Further let $|X\cap Y|=s$ and $|V-(X\cup Y)|=t$. Then $m+n-s=p-t$, hence $mn=m(p-m+s-t)$. Using elementary calculus and the fact that m, n are integral, we have

$$mn\leqslant\left\lfloor\left(\frac{p+(s-t)}{2}\right)^2\right\rfloor. \tag{2}$$

Hence if $s-t\leqslant 1$, then $mn\leqslant\left\lfloor\left(\frac{p+1}{2}\right)^2\right\rfloor=\left\lceil\frac{p^2+2p}{4}\right\rceil$ and the result is true in this case.

We now show that the situation $s-t\geqslant 2$ is impossible. Suppose not and let $X\cap Y=\{x_1, x_2, \ldots, x_s\}$. Our assumption implies $s\geqslant 2$. Either $G[X\cap Y]$ or $\bar{G}[X\cap Y]$ has no isolated vertex. Without losing generality suppose that $G[X\cap Y]$ has this property.

Since X is an irredundant set of G, it follows from (1) that for each $i=1, \ldots, s$, there exists a vertex $f(x_i)$ such that

$$f(x_i)\in N_G(x_i)\cap(V-X) \quad\text{and}\quad N_G(f(x_i))\cap X=\{x_i\}. \tag{3}$$

Further, this definition implies that $f(x_1), f(x_2), \ldots, f(x_s)$ are distinct and since $s>t$, one of these, say $f(x_s)$, belongs to $Y-X$.

By (3) $f(x_s)$ is adjacent in $\bar{G}$ to each vertex of $X-Y$ and to each x_i, $i=1, \ldots, s-1$. Therefore, $x_1, \ldots, x_{s-1}$ are not isolated vertices in $\bar{G}[Y]$. Since Y is an irredundant set of $\bar{G}$, it follows from (1) that for each $i=1, \ldots, s-1$ there exists a vertex $g(x_i)$ such that

$$g(x_i)\in N_{\bar{G}}(x_i)\cap(V-Y) \quad\text{and}\quad N_{\bar{G}}(g(x_i))\cap Y=\{x_i\}. \tag{4}$$

This definition implies that $g(x_1), \ldots, g(x_{s-1})$ are distinct. Further, no $g(x_i)$ belongs to $X - Y$, for otherwise $N_{\bar{G}}(g(x_i)) \cap Y$ would contain $f(x_s)$, contrary to (4). It follows that $g(x_1), \ldots, g(x_{s-1})$ are in $V - (X \cup Y)$, which gives $t \geqslant s - 1$, a contradiction.

Proof (b). Let G attain the bound of the theorem. It is immediate from the proof above that $s - t = 1$. Moreover, the bound of (2) is attained only when (m, n) or (n, m) is equal to $\left(\left\lfloor \frac{p+1}{2} \right\rfloor, \left\lceil \frac{p+1}{2} \right\rceil\right)$. It will now be shown that $s = 1$ and $t = 0$.

Without loss of generality assume as before that $G[X \cap Y]$ has no isolated vertices. We introduce the mappings f (satisfying (3) with $f(x_s)$ in $Y - X$) and g (satisfying (4)) exactly as in the proof of (a). As before, no $g(x_i)$ belongs to $X - Y$. Since $t = s - 1$,

$$\{g(x_1), \ldots, g(x_{s-1})\} = V - (X \cup Y).$$

Firstly suppose $s \geqslant 3$. Then by definition of the $g(x_i)$ each vertex of $V - (X \cup Y)$ is adjacent in G to at least two vertices of $X \cap Y$. Therefore $f(x_1)$ cannot be in $V - (X \cup Y)$ and hence $f(x_1) \in Y - X$. Secondly if $s = 2$, then clearly $f(x_1) \neq g(x_1)$. Therefore $f(x_1) \in Y - X$ in this case also. The definition of $f(x_1)$ implies that $f(x_1)x_s$ is an edge of $\bar{G}$. Therefore x_s is not isolated in $\bar{G}[Y]$ and so there exists a vertex $g(x_s)$ such that (4) holds with $i = s$. Then $g(x_s)$ must belong to $X - Y$ and hence is adjacent in $\bar{G}$ to $f(x_s)$, a contradiction.

We now prove that $G(X)$ has no edges. A similar argument shows that $\bar{G}[Y]$ has no edges, i.e. that $G[Y]$ is complete. Let $X \cap Y = \{x_1\}$ and assume first that x_1 is not isolated in $G[X]$. Consider $x_2 \in N_G(x_1) \cap X$. There exists a vertex $f(x_2) \in Y - X$ such that (3) holds with $i = 2$. Since $x_1 f(x_2)$ is an edge of $\bar{G}$, x_1 is not isolated in $\bar{G}[Y]$ and therefore there exists a vertex $g(x_1) \in X - Y$ such that (4) holds with $i = 1$. But then $f(x_2)$ and $g(x_1)$ are non-adjacent in both G and $\bar{G}$ which is impossible.

Thus x_1 is isolated in $G[X]$. Similarly, x_1 is isolated in $\bar{G}[Y]$. Then if x_2 is a non-isolated vertex of $G[X]$, there exists a vertex $f(x_2)$ such that (3) holds with $i = 2$, contradicting the fact that all vertices in $Y - \{x_1\}$ are adjacent in G to x_1.

We have proved that if G attains the bound, then

(i) $V = X \cup Y$ where $|X \cap Y| = 1$ and (m, n) or (n, m) is equal to

$$\left(\left\lfloor \frac{p+1}{2} \right\rfloor, \left\lceil \frac{p+1}{2} \right\rceil\right).$$

(ii) $G[X]$ and $\bar{G}[Y]$ have no edges.

Conversely any graph G satisfying (i) and (ii) has X, Y independent and hence irredundant in G, $\bar{G}$, respectively and hence $\mathrm{IR}(G) \cdot \mathrm{IR}(\bar{G}) \geqslant \left\lceil \frac{p^2 + 2p}{4} \right\rceil$. Therefore G attains the bound. This completes the proof. □

3. Consequences

Corollary 1. *For any graph G,* $\mathrm{IR}(G)+\mathrm{IR}(\bar{G})\leqslant p+1$.

Proof. In the proof of Theorem 1(a) we have $m+n-s=p-t$ and $s-t\leqslant 1$, hence the result. We note that K_p attains this bound. □

Let $\gamma(G)$ and $\Gamma(G)$ ($i(G)$ and $\beta(G)$) denote the smallest and largest cardinalities of a minimal dominating (maximal independent) vertex subset of G. We abbreviate $\gamma(G)$ by γ, $\gamma(\bar{G})$ by $\bar{\gamma}$ etc. It is well known that for any graph G,

$$\gamma\leqslant i\leqslant\beta\leqslant\Gamma\leqslant\mathrm{IR}. \tag{5}$$

Jaeger and Payan [5] proved that $\gamma\bar{\gamma}\leqslant p$ for any p-vertex graph, while Chartrand and Schuster [1] proved that $\beta\bar{\beta}\leqslant\left\lceil\frac{p^2+2p}{4}\right\rceil$, a result which can be obtained as a corollary to Theorem 1.

Corollary 2. *For any p-vertex graph G, the products $i\bar{i}$, $\gamma\bar{\beta}$, $\beta\bar{\beta}$, $\gamma\bar{\Gamma}$, $\Gamma\bar{\Gamma}$ (for example) are all bounded above by* $\left\lceil\frac{p^2+2p}{4}\right\rceil$.

Proof. Immediate from the theorem and (5). The extremal graphs of Theorem 1(b) in which $S=\emptyset$, attain the upper bound of Corollary 2 for all of the products except $i\bar{i}$. □

It is easily verified that all the graphs described in Theorem 1(b) have $i\bar{i}<\mathrm{IR}\cdot\bar{\mathrm{IR}}$. Therefore for any p-vertex graph, $i\bar{i}<\left\lceil\frac{p^2+2p}{4}\right\rceil$.

Let p be divisible by 4 and let $V(G)$ be partitioned into sets of equal size X, Y where $G[X]$ is edge free, $G[Y]$ is complete and the bipartite graph induced by X, Y is regular of degree $p/4$. Then $i=\bar{i}=\frac{p}{4}+1$ and hence $i\bar{i}=\frac{(p+4)^2}{16}$.

Corollary 3. *The maximum value $h(p)$, of $i\bar{i}$ among p-vertex graphs G satisfies*

$$\frac{(p+4)^2}{16}\leqslant h(p)<\left\lceil\frac{p^2+2p}{4}\right\rceil.$$

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